MANIPAL INSTITUTE OF TECHNOLOGY
BENGALURU
(A constituent unit of MAHE, Manipal)

I Semester MIDTERM TEST (Chemistry Cycle)
Computational Mathematics -I (MAT_1172) Answer Key

Q. No	Answer
1	$k = 1,2$
2	2
3	the system has unique solution
4	$P = \begin{bmatrix} -2 & -\frac{5}{2} \\ 1 & 1 \end{bmatrix}$
5	B
6	Linearly independent
7	$nullity(T) = 2$
8	$a = -6, b = 1, c = 4$
9	$y = c_1e^{-3x} + c_2e^{-4x}$
10	$Ae^{-x} + BCosx + CSinx$

11. aUsing Gram-Schmidt orthogonalization process construct an orthonormal vectors from the set of vectors $\{\langle 1,-1,1\rangle, \langle 1,0,1\rangle, \langle 1,1,2\rangle\}$ from $\mathbb{R}^3$.

Solution:

Let $a_1 = \langle 1,-1,1\rangle, a_2 = \langle 1,0,1\rangle, a_3 = \langle 1,1,2\rangle$

$v_1 = a_1 = \langle 1,-1,1\rangle$ ---------------(0.5)

$v_2 = a_2 - \frac{\langle a_2,v_1\rangle}{\|v_1\|^2}.v_1 = \langle \frac{1}{3},\frac{2}{3},\frac{1}{3}\rangle$ ---------------(1)

$v_3 = a_3 - \frac{\langle a_3,v_1\rangle}{\|v_1\|^2}.v_1 - \frac{\langle a_3,v_2\rangle}{\|v_2\|^2}.v_2 = \langle -\frac{1}{2}, 0, \frac{1}{2}\rangle$ ---------------(1)

The orthonormal basis is

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\langle 1,-1,1\rangle$ ---------------(0.5)

$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}\langle 1,2,1\rangle$ ---------------(0.5)

$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}\langle -1,0,1\rangle$ ---------------(0.5)

11.b) Determine the Inverse of a matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ by Gauss Jordan Method.

[A/I]= $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \qquad R_3 \leftarrow R_3 - R_1 \qquad \text{------(1)}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \qquad R_1 \leftarrow R_1 - R_2 \qquad \text{------(1)}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{1}{2} \end{array}\right] \qquad R_3 \leftarrow R_3 \div 2 \qquad \text{------(1)}$$

11.c) Solve by Gauss-Jacobi iteration method: $6x + 2y - z = 4, x + 5y + z = 3, 2x + y + 4z = 27$ to obtain the solution up to three decimal points of accuracy. Perform three iterations only.

Solution: Diagonally Dominant $x = \frac{1}{6}(4 - 2y + z), y = \frac{1}{5}(3 - x - z)\ , z = \frac{1}{4}(27 - 2x - y)$ --------(1)

Iteration	x	y	z	
1	0.667	0.6	6.75	-------(1)
2	1.592	-0.883	6.267	-------(0.5)
3	2.006	-0.972	6.175	-------(0.5)

12. a) Find all the eigen values of the matrix $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ and one of the eigen vectors of the matrix.

Solution:

Characteristics Matrix

$$\boldsymbol{A - xI} = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - x\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-x & 2 & -2 \\ -5 & 3-x & 2 \\ -2 & 4 & 1-x \end{bmatrix}$$

Characteristics polynomial and Equation:

$$|A - xI| = \begin{vmatrix} 4-x & 2 & -2 \\ -5 & 3-x & 2 \\ -2 & 4 & 1-x \end{vmatrix} = -x^3 + 8x^2 - 17x + 10 \text{ and}$$

$|A - xI| = 0$, implies $-x^3 + 8x^2 - 17x + 10 = 0$ **(1.5 Marks)**

Characteristics values/ Eigen values of A

$$x^3 - 8x^2 + 17x - 10 = 0$$

Or $(x-1)(x-2)(x-5)=0$

Or $x = 1, 2, \& \ 5$ **(1 Marks)**

Eigen Vector Corresponding to $x = 1$	**Eigen Vector Corresponding to $x = 2$**	**Eigen Vector Corresponding to $x = 5$**	**Marks for any one**
$(A-1I)X_1 = 0$ Or $\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Now, reducing the coefficient matrix to Echelon form, we get $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ On simplifying, we get $x_1 - \frac{x_3}{2} = 0 \Longrightarrow x_1 = \frac{x_3}{2}$, $x_2 - \frac{x_3}{4} = 0 \Longrightarrow x_2 = \frac{x_3}{4}$, and also $x_3 = x_3$ Thus, $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{4} \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}$ Therefore, eigenvector corresponding to the eigen value $x = 1$ is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}$ (after letting $x_3 = 1$)	$(A-2I)X_2 = 0$ Or $\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ----------- **(0.5 Marks)** Now, reducing the coefficient matrix to Echelon form, we get $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ On simplifying, we get $x_1 - \frac{x_3}{2} = 0 \Longrightarrow x_1 = \frac{x_3}{2}$, $x_2 - \frac{x_3}{2} = 0 \Longrightarrow x_2 = \frac{x_3}{2}$, and also $x_3 = x_3$ Thus, $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{x_3}{2} \\ \frac{x_3}{2} \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ Therefore, eigenvector corresponding to the eigen value $x = 2$ is $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$ (after letting $x_3 = 1$)	$(A-5I)X_3 = 0$ Or $\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ --------- --**(0.5 Marks)** Now, reducing the coefficient matrix to Echelon form, we get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ On simplifying, we get $x_1 = 0$; , $x_2 - x_3 = 0 \Longrightarrow x_2 = x_3$, and also $x_3 = x_3$ Thus, $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ Therefore, eigenvector corresponding to the eigen value $x = 5$ is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (after letting $x_3 = 1$)	**(0.5 Marks)** **(1 Mark)**

12. b) Prove that a subset of a vector space is either Linearly Independent or one of the vectors can be expressed as a linear combination of preceding vectors..

Proof: Let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of V.
If $v_1, v_2, \cdots, v_n$ are linearly independent then there is nothing to prove.
Suppose that $v_1, v_2, \cdots, v_n$ are linearly dependent.
Then there exist scalars α_i, $1 \le i \le n$ (not all zeros) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0. \quad (1)$$

----(1mark)

Since all α_i are not zero, there exists a largest positive integer k such that $\alpha_k \neq 0$.

Then $\alpha_{k+1} = 0, \ldots, \alpha_n = 0$.

So ① $\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_k v_k = 0$

------(1mark)

$$\Rightarrow \alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \cdots - \alpha_{k-1} v_{k-1}$$

$$\Rightarrow v_k = \left(-\frac{\alpha_1}{\alpha_k}\right) v_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) v_2 + \cdots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) v_{k-1}$$

where $\left(-\frac{\alpha_i}{\alpha_k}\right) \in F, \ 1 \leq i \leq k-1$.

Hence v_k is a linear Combination of v_i, $1 \leq i \leq k-1$. --(1mark)

12.c) $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x - 3y - 2z, y - 4z, z)$ for all $(x, y, z) \epsilon R^3$. Then prove that T is an invertible linear operator on $\mathbb{R}^3$. Also, find $T^{-1}(1,2,3)$.

12.c. $T(\alpha u + \beta v) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$

$= [\alpha x_1 + \beta x_2 - 3\alpha y_1 - 3\beta y_2 - 2\alpha z_1 - 2\beta z_2,$

$\alpha y_1 + \beta y_2 - 4\alpha z_1 - 4\beta z_2, \alpha z_1 + \beta z_2)$

$= \alpha[x_1 - 3y_1 - 2z_1, y_1 - 4z_1, z_1] + \beta[x_2 - 3y_2 - 2z_2, y_2 - 4z_2, z_2]$

$= \alpha T(u) + \beta T(v)$ —— (1)

-----(1mark)

$T(u) = 0$

$T(x, y, z) = (x - 3y - 2z, y - 4z, z) = (0,0,0)$

$z = 0$

$x - 0 - 0 = 0$, $y - 0 = 0$

$x = 0$, $y = 0$

$\therefore T(0) = 0$.

$\therefore T_N = \{0\}$ T is non-singular.

$\Rightarrow$ T is Invertible. (1)

------(1mark)

$T^{-1}(1,2,3)$:

$T(x,y,z) = (x-3y-2z,\ y-4z,\ z) = (1,2,3)$

$x-3y-2z=1$ | $y-4z=2$ | $z=3$ | $x-3(14)-2(3)=1$

$y = 2+4(3) = 14$ | $x-42-6=1$

$x = 1+42+6 = 49$

$\therefore T^{-1}(1,2,3) = (49,14,3)$ — (1)

---(1mark)

*********The End***************