Scheme of Evaluation

MAT 1172 - Computational Mathematics I

11. Char. eqⁿ is $\left.\begin{array}{l}\lambda^2 - 5\lambda + 6 = 0 \\ \Rightarrow \lambda = 2, 3\end{array}\right\}$ — 1M

For $\lambda = 2$; eigen vector is, $X_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For $\lambda = 3$; eigen vector is $X_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ 1½M

Matrix $P = (X_1\ X_2) = \begin{pmatrix} -1 & -1/2 \\ 1 & 1 \end{pmatrix}$

Diagonal matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ — ½M

$$P^{-1} = \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix}$$

$\therefore$ Req'd diagonalization is, 1M

$$A = \begin{pmatrix} -1 & -1/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix}$$

OR

$$A = \begin{pmatrix} -1/2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -2 & -1 \end{pmatrix}$$

12. Let $\alpha v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

then

$$(\alpha_1 + 2\alpha_2 + 2\alpha_3,\ 5\alpha_2 + 7\alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3) = (0,0,0)$$

$$\Rightarrow \begin{aligned} \alpha_1 + 2\alpha_2 + 2\alpha_3 &= 0 \\ 5\alpha_2 + 7\alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

$\therefore$ determinent of the coeff. matrix

$$= \begin{vmatrix} 1 & 2 & 2 \\ 0 & 5 & 7 \\ 1 & 1 & 1 \end{vmatrix} = 2 \neq 0$$

$1\frac{1}{2}$m

$\therefore$ B is a linearly independent set of 3 vectors in $\mathbb{R}^3$.

Since every linearly independent set of n vectors from an n-dimensional vector space V forms a basis for V we've, B forms a basis for $\mathbb{R}^3$. $\frac{1}{2}$m

Let $(4,5,3) = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ then

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 2\alpha_3 &= 4 \\ 5\alpha_2 + 7\alpha_3 &= 5 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 3 \end{aligned}$$

1m

$$\Rightarrow \alpha_1 = 2, \quad \alpha_2 = 1 \quad \alpha_3 = 0$$

$$\therefore (4,5,3) = \underline{\underline{2v_1 + 1\cdot v_2 + 0\cdot v_3}}$$

13. Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_i v_i + \cdots + \alpha_n v_n = 0$ — ½M

we know, $\langle 0, v_i \rangle = 0 \;\; \forall\, i = 1, 2 \cdots n$ — ½M

$$\Rightarrow \langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_i v_i + \cdots + \alpha_n v_n, v_i \rangle = 0$$

$$\Rightarrow \langle \alpha_1 v_1, v_i \rangle + \langle \alpha_2 v_2, v_i \rangle + \cdots + \langle \alpha_i v_i, v_i \rangle + \cdots + \langle \alpha_n v_n, v_i \rangle = 0$$

$$\Rightarrow \alpha_1 \langle v_1, v_i \rangle + \alpha_2 \langle v_2, v_i \rangle + \cdots + \alpha_i \langle v_i, v_i \rangle + \cdots + \alpha_n \langle v_n, v_i \rangle = 0$$

1M

$$\Rightarrow \alpha_1 \cdot 0 + \alpha_2 \cdot 0 + \cdots + \alpha_i \|v_i\|^2 + \cdots + \alpha_n \cdot 0 = 0$$

$\Rightarrow \alpha_i \|v_i\|^2 = 0$ for $i = 1, 2, \cdots n$

$\Rightarrow \alpha_i = 0$ or $\|v_i\|^2 = 0$ for $i = 1, 2 \cdots n$

½M

But $\|v_i\|^2 = 0 \Rightarrow v_i = 0$, impossible.

$\therefore \alpha_i = 0$ for $i = 1, 2, \cdots n$

$\therefore \{v_1, v_2, \cdots v_n\}$ $\underline{\underline{\text{is L.I.}}}$

½M

14. Augmented matrix

$$(A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & -1 & 2 & 6 \\ 3 & 5 & -7 & 14 \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & -2 & 1 & -2 \\ 0 & 2 & -10 & -10 \end{array}\right) \quad R_2 \to \frac{R_2}{-2}$$

——— ½M

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 2 & -10 & -10 \end{array}\right) \quad R_3 \to R_3 - 2R_2$$

——— ½M

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & -9 & -12 \end{array}\right), \text{ Echelon form}$$

——— ½M

$\therefore \rho(A|B) = \rho(A) = 3 =$ no. of unknowns

$\therefore$ The system is consistent and has unique sol^n.

——— ½M

$\therefore$ Equivalent matrix eq^n is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -12 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} x+y+z &= 8 \\ y-\frac{z}{2} &= 1 \\ -9z &= -12 \end{aligned} \quad \Rightarrow \quad \begin{aligned} z &= \frac{4}{3} \\ y &= 5/3 \\ x &= 5 \end{aligned} \quad \Bigg\} \text{1M}$$

15. $$\left.\begin{aligned} x &= \frac{1}{5}(12-2y-z) \\ y &= \frac{1}{4}(15-x-2z) \\ z &= \frac{1}{5}(20-x-2y) \end{aligned}\right\} \tfrac{1}{2}\text{M}$$

$$x^{(0)} = y^{(0)} = z^{(0)} = 0$$

Iteration	$x^{(n)}$	$y^{(n)}$	$z^{(n)}$	
1	2.4	3.15	2.26	— ½M
2	0.688	2.448	2.8832	— 1M
3	0.8442	2.0974	2.9922	— ½M

$\therefore$ After three iterations approx. sol^n correct to 3 decimal places is
$x = 0.844$; $y = 2.097$; $z = 2.992$ } ½M

16. <u>Step 1</u> :- Let $v_1 = a_3 = (0,0,1)$ then

$$u_1 = \frac{v_1}{\|v_1\|} = (0,0,1)$$ 1M

<u>Step 2</u> :- $v_2 = a_2 - \langle a_2, u_1 \rangle u_1$

$\langle a_2, u_1 \rangle = 2$

$\therefore v_2 = (-1,0,2) - 2(0,0,1)$ 1M

$= (-1,0,0) \quad \therefore u_2 = \frac{v_2}{\|v_2\|} = (-1,0,0)$

<u>Step 3</u> :- $v_3 = a_1 - \langle a_1, u_1 \rangle u_1 - \langle a_1, u_2 \rangle u_2$

$\Rightarrow \langle a_1, u_1 \rangle = 2, \quad \langle a_1, u_2 \rangle = -1$

$\therefore v_3 = (1,2,2) - 2(0,0,1) + 1(-1,0,0)$ 1M

$= (0,2,0) \quad \therefore u_3 = \frac{v_3}{\|v_3\|} = (0,1,0)$

$\therefore$ Req'd orthonormal basis is

$$\{(0,0,1), (-1,0,0), (0,1,0)\}$$

17. Let $u = (x_1, y_1)$, $v = (x_2, y_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$ then

$$\alpha u + \beta v = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \in \mathbb{R}^2$$ ½m

$$\therefore T(\alpha u + \beta v) = T(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$= \left(2(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2), 3(\alpha y_1 + \beta y_2)\right)$$

$$= \left(\alpha(2x_1 + y_1) + \beta(2x_2 + y_2), \alpha(3y_1) + \beta(3y_2)\right)$$

$$= \left(\alpha(2x_1 + y_1), \alpha(3y_1)\right) + \left(\beta(2x_2 + y_2), \beta(3y_2)\right)$$

$$= \alpha(2x_1 + y_1, 3y_1) + \beta(2x_2 + y_2 + 3y_2)$$

$$= \alpha Tu + \beta Tv$$ 1½m

$\therefore$ T is a L.T

18. $T(1,0) = (1,1,1)$

$$= c_1(1,1,0) + c_2(1,-1,0) + c_3(0,0,1)$$

$$= (c_1 + c_2, c_1 - c_2, c_3)$$

$\therefore c_3 = 1, \quad \begin{matrix} c_1 - c_2 = 1 \\ c_1 + c_2 = 1 \end{matrix} \Rightarrow c_1 = 1, c_2 = 0$

$\therefore T(1,0) = 1(1,1,0) + 0(1,-1,0) + 1.(0,0,1)$ — 1M

$T(0,1) = (1,-1,1)$

$= (c_1 + c_2, c_1 - c_2, c_3)$

$\therefore c_3 = 1, \quad \begin{matrix} c_1 - c_2 = -1 \\ c_1 + c_2 = 1 \end{matrix} \Rightarrow 2c_1 = 0 \Rightarrow c_1 = 0$

$\therefore c_2 = 1$

$\therefore T(0,1) = 0.(1,1,0) + 1.(1,-1,0) + 1.(0,0,1)$ — 1/2M

$$\therefore [T] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$ — 1/2M

19. Auxillary eq^n is, $m^3 + 6m^2 + 11m + 6 = 0$

$\Rightarrow m = -1, -2, -3$ } 1M

$\therefore y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ — 1M